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ELECTROMAGNETIC WAVE PROPAGATION IN BOUNDED ELECTRON BEAMS

by

PHILIP PARZEN

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ABSTRACT

The propagation of electromagnetic waves in a region containing free electrons is discussed. The equations are linearized by making the small signal approximation. First, it is shown that these linearized equations will have a unique solution in case the time-varying current is parallel to the velocity if the tangential components of either the magnetic or electric field and the normal component of the current are specified on the surface bounding the region. Second, these equations are used to discuss the propagation of electromagnetic waves in a circular cylinder completely filled with electrons. It is shown how to obtain a convergent expansion for the field quantities. This problem may have applications to drift tubes.

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I. INTRODUCTION AND STATEMENT OF PROBLEM

The propagation of sinusoidal electromagnetic waves in charge-free space, guided or unbounded, has been studied in great detail(1). The purpose of this report is to extend the study to the guided propagation of electromagnetic waves in regions containing charged particles, such as an electron beam. Although the equations for charge free space are linear, the equations for electron beams are nonlinear. A similar study in the case of electron beams is possible only upon linearization of the equations. The linearization is accomplished by decomposing each field quantity into a sum of a time-independent term and of a relatively much smaller sinusoidal term. After substituting this into the original nonlinear equations, it is assumed that terms of higher order such as the product of two sinusoidal terms, may be neglected. The resulting linear equations govern the variation in space of the electric and magnetic fields, and, in addition, the variation of those field quantities which characterize the electron beam, namely, electron current density, velocity, and charge density.

We shall study the extensions of the two following problems in charge free space. First is the uniqueness problem for charge-free space. It is well known(2) that the fields within a multiply connected region are uniquely determined by either the component of the electric field or the component of the magnetic field which is tangent to the surface bounding the region. A similar uniqueness theorem is proven for the sinusoidal propagation of electromagnetic waves in electron beams. The theorem states that, under certain conditions, the sinusoidal field quantities are determined uniquely within a multiply connected region by specifying on the surfaces bounding the region the following quantities: the component of the sinusoidal electric field tangent to the surfaces, and the sinusoidal electron velocity, or the sinusoidal current density normal to the surfaces.

The second problem is to determine the modes of propagation of sinusoidal electromagnetic waves in a right cylinder whose lateral surface is of infinite conductivity. The extension of this problem to be studied is that in which the right cylinder is completely filled with an electron beam, whose time-independent velocity is a function only of the axial distance. It is also assumed that the current density and electron velocity possess only axial components. The determination of the modes of propagation depends upon the solution of a system of two second-order ordinary differential equations, which are coupled to each other. These equations are solved by the matrizant(3) method in the form of an infinite series. We prove the convergence of this series. The necessary boundary conditions are specified in accordance with the uniqueness theorem. The results derived by this method may be applied to the design of microwave electron tubes, for which a knowledge of the modes of propagation is indispensable.

II. FORMULATION OF EQUATIONS AND PROOF OF UNIQUENESS THEOREM

The basic Maxwell equations(1) in MKS units are:

$$\nabla \times \bar{E} = -\mu_0 \bar{H}_t \tag{1}$$

$$\nabla \times \bar{H} = \bar{J} + \sigma\bar{E} + \epsilon_0 \bar{E}_t \tag{2}$$

$$\nabla \cdot \bar{J} + \bar{\rho}_t = 0 \tag{3}$$

$$\frac{d\bar{V}}{dt} = \frac{e}{m}\left[\bar{E} + \bar{V} \times \bar{H}\right] \tag{4}$$

$$\bar{J} = \bar{\rho}\,\bar{V} \tag{5}$$

Here, $\bar{E}$, $\bar{H}$, $\bar{J}$, $\bar{V}$, and $\bar{\rho}$ denote the electric field, magnetic field, current density, electron velocity, and charge density, respectively. ϵ_0, μ_0, σ, $-e$, and m are the dielectric constant, permeability, conductivity of the medium, charge and mass of the electron, respectively, which are assumed to be constants and positive real numbers. The subscript t denotes partial differentiation with respect to time, and $\frac{d}{dt}$ denotes the total derivative. Thus

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{V} \cdot \tag{6}$$

Capital letters will denote vectors in this section, unless it is specified otherwise.

In accordance with the linearization process, each field quantity is written as the sum of a time-independent term and a relatively much smaller sinusoidal term. Thus, for example,

$$\bar{J} = J_0 + J \exp(i\omega t) \; ; \; i = (-1)^{\frac{1}{2}} \tag{7}$$

$$|J| \ll |J_0|$$

Here ω is the angular frequency which is a positive real number. The exponential function is introduced to simplify the resultant linear equations. The physical solution is obtained by taking the real part of the right-hand side of equation (7). Substituting formulas similar to (7) into the original equations and neglecting higher order terms, we obtain the following equations after separating the time-independent and the time-dependent terms.

A. Time-Dependent Equations

$$\nabla \times E = -i\,\omega\mu_0 H \tag{1a}$$

$$\nabla \times H = J + \sigma E + i\omega\epsilon_0 E \tag{2a}$$

$$\nabla \cdot J + i\omega\rho = 0 \qquad (3a)$$

$$i\omega V + V_0 \cdot \nabla V + V \cdot \nabla V_0 = \frac{e}{m}\left[E + V \times H_0 + V_0 \times H\right] \qquad (4a)$$

$$J = \rho_0 V + \rho V_0 \qquad (5a)$$

B. Time-Independent Equations

$$\nabla \times E_0 = 0 \qquad (1b)$$

$$\nabla \times H_0 = J_0 + \sigma E_0 \qquad (2b)$$

$$\nabla \cdot J_0 = 0 \qquad (3b)$$

$$V_0 \cdot \nabla V_0 = \frac{e}{m}\left[E_0 + V_0 \times H_0\right] \qquad (4b)$$

$$J_0 = \rho_0 V_0 \qquad (5b)$$

It is assumed that the time-independent field quantities have already been obtained from equations (1b) - (5b) and hence are to be considered as known quantities. The problem is to solve equations (1a) - (5a) for the sinusoidal field quantities subject to boundary conditions which are chosen in accordance with a uniqueness theorem. In view of the complexity of the equations, it has not been possible to obtain a uniqueness theorem for all cases. However for the special case in which J and V are parallel, the following uniqueness theorem is proved.

Theorem I

If J and V are parallel within a multiply connected region G, the sinusoidal field quantities which are bounded and possess continuous first derivatives are determined uniquely within G by specification of the following quantities on the boundary surfaces S: $E \times \underline{n}$ or $H \times \underline{n}$, and V or $J \cdot \underline{n}$. $\underline{n}$ is the unit normal to S pointing away from the region G.

Proof

From equations (1a), (2a) and the use of Gauss' integral theorem, it follows that

$$\int (H^* \cdot \nabla \times E - E \cdot \nabla \times H^*)\, d\gamma = \int \nabla \cdot (E \times H^*)\, d\gamma = \int E \times H^* \cdot ds$$

$$= i\omega \int \mu_0\, H \circ H^*\, d\gamma - i\omega \int \epsilon_0\, E \circ E^*\, d\gamma + \int \sigma E \circ E^*\, d\gamma + \int J^* \circ E\, d\gamma . \quad (6)$$

Here $d\gamma$ is an element of volume within G and the asterisk denotes the complex conjugate. Using the vector equation identity,

$$\nabla (V_0 \circ V) = V_0 \circ \nabla V + V \circ \nabla V_0 + V_0 \times \nabla \times V + V \times \nabla \times V_0 , \quad (7)$$

equation (4a) may be rewritten as follows:

$$i\omega V + \nabla (V_0 \cdot V) = \frac{e}{m}\left[E + V \times H_0 + V_0 \times H \right] - V_0 \times \nabla \times V - V \times \nabla \times V_0 . \quad (8)$$

Also since J and V are parallel, J, V, J_0, and V_0 are parallel to each other. Hence taking the scalar product of equation (8) with J^*,

$$J^* \circ E = \frac{m}{e}\left[i\omega V \circ J^* + J^* \circ \nabla (V_0 \circ V) \right] . \quad (9)$$

Using equations (3a) and (5a),

$$i\omega V \cdot J^* = i\omega \rho_0\, V \circ V^* + \nabla \cdot (V_0 \circ V)\, J^* - J^* \circ \nabla (V_0 \circ V) . \quad (10)$$

Substituting (10) into equation (6), one obtains

$$i\omega \int \left[\mu_0 H \cdot H^* - \epsilon_0 E \cdot E^* + \frac{m}{e} \rho_0 V \circ V^* \right] d\gamma + \int \sigma E \circ E^*\, d\gamma$$

$$+ \int \left[E \times H^* + (V_0 \cdot V)\, J^* \right] \circ dS = 0 . \quad (11)$$

Equation (11) may now be used to prove the uniqueness theorem. Let us assume that there are two field solutions satisfying the same boundary conditions mentioned in the theorem. The difference of these two solutions, which is also a solution because of the linearity of the equations, will have corresponding boundary values equal to 0. Hence for the difference solution, the surface integral in equation (11) will vanish. Equating real and imaginary parts in equation (11), it follows that

$$\int \sigma E \cdot E^* \, d\tau = 0, \qquad (12)$$

which is possible only if $E = 0$, since the integrand is positive definite. The vanishing of the other field components follows directly from the linearized equations. That the solution is unique is an immediate consequence of the vanishing of a solution with zero boundary conditions.

In the case of $\sigma = 0$, it can be deduced only that

$$\int \left\{ \mu_0 H \cdot H^* - \epsilon_0 E \cdot E^* + \frac{m}{e} \rho_0 V \cdot V^* \right\} d\tau = 0 . \qquad (13)$$

This equation can be satisfied with non-vanishing fields only at certain discrete frequencies, which are the resonant frequencies of the region G in the presence of an electron beam. In this case, the uniqueness theorem will hold at all frequencies except those which are resonant.

The uniqueness theorem has been proven only for field solutions in which J is parallel to V. An important question is whether there exist non-trivial cases in which the assumption prevails. This case occurs when H_0 is infinite. Thus, if it is assumed that V_0 is bounded and possesses bounded derivatives, it follows from equations (4a) and (4b) that both V and V_0 are parallel to H_0. Hence J is parallel to V. This case is discussed in greater detail subsequently.

III. MODES OF PROPAGATION IN A RIGHT CYLINDER COMPLETELY FILLED WITH AN ELECTRON BEAM

The problem to be considered is to obtain solutions of the linearized equations within a right cylinder of arbitrary cross section between the planes $z = z_o$, z_1 (Fig. 1).

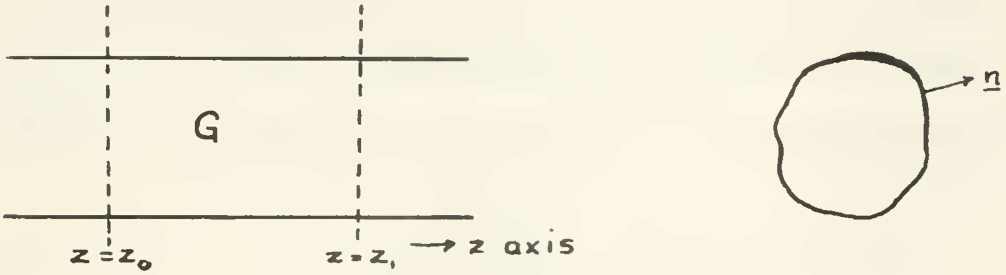

Fig. 1

The lateral surface is assumed to be of infinite conductivity and H_o is assumed to be infinite and along the z axis. On the planes $z = z_o$, z_1, $E \times \underline{z}$ and J are given; $\underline{z}$ being the unit vector along the z axis. Another assumption is that V_o is an analytic function of z only and does not vanish anywhere in the interval $z_o \leq z \leq z_1$. As shown previously, J, V, V_o, and J_o are parallel to the Z axis. Thus the boundary conditions in the region G are:

at $z = z_o$: $E \times \underline{z}$ and $J = E_s(0)$ and $J(0)$, respectively; (14)

$z = z_1$: $E \times \underline{z}$ and $J = E_s(1)$ and $J(1)$, respectively; (15)

on the lateral surface: $E \times \underline{n} = 0 \qquad J \cdot \underline{n} = 0$. (16)

Here $\underline{n}$ denotes the unit vector normal to the boundary curve of the cross section and the subscript s denotes components of the vector in the transverse plane.

It follows from the uniqueness theorem that there exists at most in the region G one solution only which satisfies these boundary conditions. Therefore, any solution which is constructed with these boundary conditions must be the only one. A solution will now be given.

The following additional notation will be used. Let J, V, V_o, J_o, E, and H denote the z components of the respective vectors. Also the subscript z will denote differentiation with respect to z. Then it follows from equations (3a) - (5a) and from the fact that J_o is constant,

$$\rho = - \frac{1}{i\omega} J_z , \tag{17}$$

$$V = \frac{V_o}{J_o} \left[J + \frac{V_o}{i\omega} J_z \right] , \tag{18}$$

$$V_o^3 J_{zz} + \left[2 i\omega V_o^3 + 3 V_o^2 V_{oz} \right] J_z + \left[2 i\omega V_o V_{oz} - \omega^2 V_o \right] J = \frac{eJ_o}{m} i\omega E . \tag{19}$$

From equations (1a) - (2a), one obtains

$$E_{zz} + \nabla_s^2 E + k^2 E = \frac{i}{\omega\epsilon} \left[k^2 J + J_{zz} \right] \tag{20}$$

$$\nabla_s \cdot E_s = - \frac{1}{i\omega\epsilon} J_z - E_z \tag{21}$$

$$k^2 E_s + E_{szz} = (\nabla_s E)_z + i\omega\mu_o (\underline{z} \times \nabla_s H) \tag{22}$$

$$H_s = - \frac{1}{i\mu_o\omega} \left[(\underline{z} \times E_s)_z - \underline{z} \times \nabla_s E \right] \tag{23}$$

$$H_{zz} + \nabla_s^2 H + k^2 H = 0 . \tag{24}$$

Here,

$$\epsilon = \epsilon_o - i \frac{\sigma}{\omega} ,$$

$$k^2 = \omega^2 \epsilon \mu_o , \tag{25}$$

and ∇_s = gradient operator in transverse plane.

With these equations, all the field quantities are expressed in terms of J, E, H, and their derivatives. It may also be shown by reversing the algebraic steps in the derivation that equations (17) - (25) are equivalent to the linearized equations if the field quantities possess continuous second derivatives.

It is convenient to split E_s and H_s into two parts(4) depending upon E and H, respectively. These are known as E modes and H modes, respectively. Let

$$E_s = E_s^{(1)} + E_s^{(2)} , \qquad (26)$$

and similarly for H_s. These quantities must satisfy the following equations:

$$\nabla_s \cdot E_s^{(1)} = - \frac{1}{i\omega\epsilon} J_z - E_z \qquad (27)$$

$$k^2 E_s^{(1)} + E_{szz}^{(1)} = (\nabla_s E)_z \qquad (28)$$

$$H_s^{(1)} = - \frac{1}{i\mu_o\omega} \left[(\underline{z} \times E_s^{(1)})_z - \underline{z} \times \nabla_s E \right] \qquad (29)$$

$$k^2 E_s^{(2)} + E_{szz}^{(2)} = i \omega\mu_o (\underline{z} \times \nabla_s H) \qquad (30)$$

$$H_s^{(2)} = - \frac{1}{i\mu_o\omega} \left[\underline{z} \times E_s^{(2)} \right]_z \qquad (31)$$

Writing each field quantity as a product of two factors depending upon coordinates in the transverse plane or z only: e.g.

$$E = E(s)\, E(z) \qquad (32)$$

the equations separate into two sets governing the dependence upon s and z respectively. These are:

A. E Mode

1. Dependence Upon s

$$
\begin{aligned}
\nabla_s^2 E + T_1^2 E &= 0 \\
J &= E \\
E_t &= \nabla_s E \qquad (33) \\
H_t &= \underline{z} \times \nabla_s E
\end{aligned}
$$

E = 0 on boundary curve of cross section. Here the variable s is understood.

2. Dependence Upon z

$$V_o^3 J_{zz} + \left[2 i \omega V_o^2 + 3 V_o^2 V_{oz}\right] J_z + \left[2 i \omega V_o V_{oz} - \omega^2 V_o\right] J = \frac{eJ_o}{m} \, i \omega E \qquad (34)$$

$$E_{zz} + (k^2 - m_1^2) E = \frac{1}{\omega \epsilon} (k^2 J + J_{zz}) \qquad (35)$$

$$- T_1^2 E_s^{(1)} = - \frac{1}{i \omega \epsilon} J_z - E_z \qquad (36)$$

$$H_s^{(1)} = - \frac{1}{i \mu_o \omega} \left[E_{sz}^{(1)} - E \right] \qquad (37)$$

Here the variable z is understood.

B. H Mode

1. Dependence Upon s

$$
\begin{aligned}
\nabla_s^2 H + T_2^2 H &= 0 \\
E_s^{(2)} &= \underline{z} \times \nabla_s H \qquad (38) \\
H_s^{(2)} &= \underline{z} \times \underline{z} \times \nabla_s H = - \nabla_s H
\end{aligned}
$$

and $\underline{z} \times \nabla_s H = 0$ on boundary curve of cross section.

2. Dependence Upon z

$$H_{zz} + (k^2 - T_2^2)\, H = 0 \tag{39}$$

$$H = \exp(\pm\, i\, h_2\, z) \quad ; \quad h_2^2 = k^2 - T_2^2 \tag{40}$$

$$E_s^{(2)} = \frac{i\mu\,\omega_o}{k^2 - h_2^2}\, H \tag{41}$$

$$H_s^{(2)} = \mp\, \frac{ih_2}{k^2 - h_2^2}\, H \tag{42}$$

The boundary conditions in equations (33) and (38) ensure that the boundary conditions in equation (16) are satisfied for the component of electric field tangent to the lateral surface. Thus the only remaining boundary conditions to be satisfied are those on $z = z_o, z_1$

Now it is well known that the E(s) and H(s), whose variations are given by equations (33) and (38) with their respective boundary conditions, exist only for a discrete set of values of T_1^2 and T_2^2, namely T_{1n}^2 and T_{2n}^2, which are real positive numbers$^{(5)}$. Furthermore, the eigenfunctions of E(s) and H(s) corresponding to different eigenvalues are orthogonal to each other and there exists an orthogonal subset of the eigenfunctions in terms of which any function of s possesses a unique convergent expansion$^{(6)}$. Similarly the corresponding $E_s^{(1)}(s)$ and $E_s^{(2)}(s)$, and $H_s^{(1)}(s)$ and $H_s^{(2)}(s)$ form a complete orthogonal set. These statements can easily be shown to be true in the case where the cross section is a rectangle or a circle by using trigonometric or Bessel eigenfunctions, respectively. It will be assumed that for other cross sections which are simply connected, it is still possible to expand $E_s(1)$, $E_s(2)$, and J(1), and J(2) in terms of transverse and axial eigenfunctions respectively and treat each term in the expansion separately. Expanding the fields in the region G in terms of these eigenfunctions, the coefficients will be functions of z whose variation is given by the mode equations with T_1^2 and T_2^2 replaced by T_{1n}^2 and T_{2n}^2. Thus let,

$$
\begin{aligned}
E &= \sum_n E_n(s)\, E_n(z) \\
J &= \sum_n E_n(s)\, J_n(z) \\
E_s &= \sum_n E_{sn}^{(1)}(s)\, E_s^{(1)}(z) + \sum_m E_{sm}^{(2)}(s)\, E_{sm}^{(2)}(z) \\
H_s &= \sum_n H_{sn}^{(1)}(s)\, H_s^{(1)}(z) + \sum_m H_{sm}^{(2)}(s)\, H_s^{(2)}(z) \\
E_s(1) &= \sum_n E_{sn}^{(1)}(s)\, a_n(1) + \sum_m E_{sm}^{(2)}(s)\, a_m(1) \\
J(1) &= \sum_n E_n(s)\, b_n(1)
\end{aligned}
\qquad (43)
$$

and similarly with $E_s(2)$ and $J(2)$. The a_n, a_m, and b_n are constants.

The coefficients for the H mode are easily obtained from the H mode expressions in equations (39) - (42). It is seen that there are two values corresponding to $h_2 = \pm h_{2m}$ to be determined. These are linear combinations of $a_m(1)$ and $a_m(2)$. They could also be determined if both E_s and H_s are specified in $z = z_o$ and nothing is specified at $z = z_1$

The coefficients for the E modes are to be obtained from the expressions for the E mode in equations (34) - (37) with the subscript n added for the respective mode. Deleting the subscript n, these constitute a set of ordinary differential equations, in which the values of $E_s(z)$ and $J(z)$ are specified at $z = z_o, z_1$. This is equivalent to solving the set of ordinary homogeneous differential equation in J and E subject to the boundary condition that the values of J and $J_z + i\omega\epsilon E_z$ are given at $z = z_o, z_1$. This constitutes a linear differential system(7) with a two-point boundary condition. The necessary and sufficient condition that there exist a unique solution is that the homogeneous differential equations with the boundary conditions J and $J_z + i\omega\epsilon E_z = 0$ at $z = z_o$, z_1 possess only the trivial solution $J = E = 0$. This has to be the case; otherwise, there would exist a solution E(s) J(z), E(s) E(z) in region G (Fig. 1) for which E_s and J are zero on the boundary, which is contrary to the uniqueness theorem. Thus the solution of the inhomogeneous boundary value problem exists and is unique. The following theorem may now be proved.

Theorem II

The fields in G exist and are uniquely determined by the boundary condition that the value of J, J_z or V, E_s and H_s be given at $z = z_o$. Nothing need be specified at $z = z_1$.

Proof

It is obvious from the expressions for the E mode in equations (34) - (37) that J, J_z, E_s and H_s at $z = z_o$ determine uniquely the values of J, J_z, E, and E_z at $z = z_o$ and conversely. Then according to the existence theorem of ordinary differential equations with analytic coefficients, there exists a unique solution which determines the value of J and E_s at $z = z_1$. Hence the field solution is unique according to the existence theorem 1. The following corollary is easily proved.

Corollary 1

The fields in region G exist and are determined uniquely by the following boundary conditions:

a) either H_s or E_s and either J or V given at $z = z_o$, z_1

b) J, J_z or V, E, and E_z given at $z = z_o$

c) J, J_z or V, E_s and H_s given at $z = z_o$

The remainder of the thesis will now be devoted to a study of the solution of the ordinary differential equations for J and E.

IV. SOLUTION OF THE DIFFERENTIAL EQUATIONS FOR J AND E

It is convenient to write the ordinary differential equations for J(z) and E(z) in matrix form.

$$U_z = A\,U \tag{44}$$

Here U is the column matrix $\begin{vmatrix} J \\ J_z \\ \mathcal{E} \\ \mathcal{E}_z \end{vmatrix}$; $\mathcal{E} = i \quad \varepsilon E$ (45)

$$A = \begin{vmatrix} 0 & 1 & 0 & 0 \\ \gamma_o^2 + i\,2T_{oz} & -i\,2\gamma_o + e\,\dfrac{\gamma_{oz}}{\gamma_o} & y^2\gamma_o^2 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2 - \gamma_o^2 - i2\,\gamma_{oz}, & i2\gamma_o - 3\,\dfrac{\gamma_{oz}}{T_o} & -k^2 + T_1^2 - y^2\gamma_o^2 & 0 \end{vmatrix} \tag{46}$$

$$\gamma_o = \frac{\omega}{V_o} \;; \qquad y^2 = \frac{e}{\omega^2{}_m}\;\frac{J_o}{\varepsilon V_o}\,. \tag{47}$$

The boundary condition is $U = U(z_o)$ at $z = z_o$. (48)

A simple solution is obtained only in the case V_o = constant and $|y^2| << 1^{(8)}$. In this case the solution may be expressed as the sum of four exponential functions with constant coefficients as follows:

$$E(z) = \sum_m E_m \exp(\lambda_m z), \quad m = 1,\ 2,\ 3,\ 4 \tag{49}$$

$$\lambda_{1,\,2} = -\,i\,\gamma_o \pm \gamma_o\;\frac{(-\gamma_o^2 + k^2)^{\frac{1}{2}}}{(-\gamma_c^2 + k^2 - T_1^2)}\;y \tag{50}$$

$$\lambda_{3,4} = \pm i\, h_1 \mp \frac{\gamma_o^2\, T_1^2}{(\pm h_1 + \gamma_o)^2} \; \frac{1}{2\, h_1} \; y^2 \qquad (51)$$

$$h_1 = (k^2 - T_1^2)^{\frac{1}{2}} . \qquad (52)$$

We consider the problem in which $|y^2|$ and $\frac{V_{oz}}{V_o}$ are both small. The previous case suggests finding a solution expressed as the sum of four exponential functions, whose coefficients are power series in y^2 and V_{oz}.

A convenient method to accomplish this is by the matrizant scheme of Peano and Baker$^{(3),(9)}$ and its extension by Keller and Keller$^{(10)}$. According to this method, the solution of the differential equation (44) is

$$U = \underset{z_o}{\overset{z}{\Omega}} \left\{ A \right\} U(z_o) \qquad (53)$$

where

$$\underset{z_o}{\overset{z}{\Omega}} \left\{ A \right\} = \text{matrizant of } A = 1 + \int_{z_o}^{z} A(x_1)dx_1 + \int_{z_o}^{z} A(x_2)dx_2 \int_{z_o}^{x_2} A(x_1)dx_1 + \cdots \qquad (54)$$

Inasmuch as $\gamma_o(z)$ is an analytic function in the region $z_o \leq z \leq z_1$, $\Omega\{A\}$ exists and converges uniformly in the region. The following properties of the matrizant will be used here:

a) $\Omega^{-1}\left\{A\right\}$ exists if $\Omega\left\{A\right\}$ exists$^{(9)}$. (55)

b) For any non-singular matrix P,

$$\underset{z_o}{\overset{z}{\Omega}} \left\{ A \right\} = P \underset{z_o}{\overset{z}{\Omega}} \left\{ P^{-1} AP - P^{-1} P_x \right\} P^{-1} (z_o)^{(10)} . \qquad (56)$$

c) For any diagonal matrix A = diag. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$,

$$\underset{z_o}{\overset{z}{\Omega}} \left\{ A \right\} = \text{diag.} \left[\exp\left(\int_{z_o}^{z} \lambda_1\, dx\right), \exp\left(\int_{z_o}^{z} \lambda_2\, dx\right), \cdots \right]^{(10)} . \qquad (57)$$

$$\text{d)}\ \overset{z}{\underset{z_0}{\Omega}} \left\{ A + B \right\} = \overset{z}{\underset{z_0}{\Omega}} \left\{ A \right\} \overset{z}{\underset{z_0}{\Omega}} \left\{ \overset{x}{\underset{z_0}{\Omega}}{}^{-1} \left\{ A \right\} B \overset{x}{\underset{z_0}{\Omega}} \left\{ A \right\} \right\}^{(9)} . \qquad (58)$$

Using equation (56), the solution is

$$U = P \overset{z}{\underset{z_0}{\Omega}} \left\{ P^{-1} A P - P^{-1} P_x \right\} P^{-1}(z_0)\, U(z_0) . \qquad (59)$$

Usually P is chosen to diagonalize A. However, in order to separate out the effects due to V_{oz}, let

$$A = A_1 + A_{2z} \qquad (60)$$

$$A_1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ \Upsilon_o^2 & -i2\Upsilon_o & y^2\Upsilon_o^2 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2-\Upsilon_o^2 & i2\Upsilon_o & -k^2+T^2-y^2\Upsilon_o^2 & 0 \end{vmatrix} \qquad (61)$$

$$A_2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ i2\Upsilon_o & 3\log\Upsilon_o & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i2\Upsilon_o & -3\log\Upsilon_o & 0 & 0 \end{vmatrix} . \qquad (62)$$

Then by equation (60)

$$U = P \overset{z}{\underset{z_0}{\Omega}} \left\{ P^{-1} A_1 P + P^{-1} A_{2x} P - P^{-1} P_x \right\} P^{-1}(z_0) U(z_0) \qquad (63)$$

where P is chosen to diagonalize A_1. Such a non-singular matrix P exists if the eigenvalues of A_1 are distinct.

The eigenvalues of A_1, λ_i, satisfy the fourth-degree equation,

$$y^2 = -\frac{(\lambda^2 + k^2 - T_1^2)\,(\lambda + i\,\gamma_o)^2}{\gamma_o^2\,(\lambda^2 + k^2)} \,. \qquad (64)$$

The roots of this equation when $|y^2|$ is small are obtained as follows: for $y = 0$, the roots $\lambda_n^{(0)}$ are:

$$\lambda_{1,2,3,4}^{(0)} = -i\,\gamma_o,\ -i\,\gamma_o,\ i\,h_1,\ -i\,h_1$$

$$h_1 = (k^2 - T_1^2)^{\frac{1}{2}}$$

Now $\frac{d(y^2)}{d\lambda}$ is not zero at $\lambda = \lambda_3^{(0)}$, $\lambda_4^{(0)}$. Hence, according to a well known theorem in the theory of functions of a complex variable, λ is an analytic function in the neighborhood of $y^2 = 0$ and the corresponding power series is

$$\lambda = \lambda_{3,4}^{(0)} + y^2 \frac{1}{\left.\frac{d(y^2)}{d\lambda}\right|_{\lambda_{3,4}^{(o)}}} + 0\,(y^4) \,. \qquad (65)$$

Hence, $$\lambda_{3,4} = \pm\, i\, h_1 \mp \frac{\gamma_o^2\, T_1^2}{(\pm\, h_1 + \gamma_o)^2}\ \frac{1}{2\,h_1}\ y^2 + 0\,(y^4) \,. \qquad (66)$$

The remaining two roots are obtained by extracting the square root of equation (43) and applying the same reasoning.

$$y = \pm \frac{(\lambda^2 + k^2 - T_1^2)^{1/2}(\lambda + i\gamma_o)}{\gamma_o(\lambda^2 + k^2)^{1/2}}$$

and $$\lambda_{1,2} = -i\gamma_o \pm \gamma_o \frac{(-\gamma_o^2 + k^2)^{\frac{1}{2}}}{(-\gamma_o^2 + k^2 - T_1^2)^{\frac{1}{2}}} \, y + O(y^2) \, . \qquad (67)$$

Thus it is seen that there is a region in the neighborhood of $y = 0$, excluding $y = 0$, in which the roots are distinct. Also let

$$(\lambda_{1,2} + i\gamma_o)^2 = g_{1,2}\, \gamma_o^2\, y^2$$

$$g_{1,2} = \frac{-\gamma_o^2 + k^2}{-\gamma_o^2 + k^2 - T_1^2} + O_{1,2}(y). \qquad (68)$$

The matrix P is obtained as follows: The elements of P or P_{mn} satisfy the equations,

$$\frac{P_{1n}}{y^2\gamma_o^2} = \frac{P_{2n}}{y^2\gamma_o^2\lambda_n} = \frac{P_{3n}}{(\lambda_n+i\gamma_o)^2} = \frac{P_{4n}}{\lambda_n(\lambda_n+i\gamma_o)^2}, \quad n = 1,2,3,4. \qquad (69)$$

Hence,

$$P = \begin{vmatrix} 1 & 1 & y^2\gamma_o^2 & y^2\gamma_o^2 \\ \lambda_1 & \lambda_2 & y^2\gamma_o^2\lambda_3 & y^2\gamma_o^2\lambda_4 \\ g_1 & g_2 & (\lambda_3+i\gamma_o)^2 & (\lambda_4+i\gamma_o)^2 \\ \lambda_1 g_1 & \lambda_2 g_2 & \lambda_3(\lambda_3+i\gamma_o)^2 & \lambda_4(\lambda_4+i\gamma_o)^2 \end{vmatrix} . \qquad (70)$$

To obtain P^{-1}, let $P = P_1 + y^2\gamma_o^2 P_2$ (71)

$$P = \begin{vmatrix} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ g_1 & g_2 & (\lambda_3+i\gamma_o)^2 & (\lambda_4+i\gamma_o)^2 \\ \lambda_1 g_1 & \lambda_2 g_2 & \lambda_3(\lambda_3+i\gamma_o)^2 & \lambda_4(\lambda_4+i\gamma_o)^2 \end{vmatrix} \qquad (72)$$

$$P_2 = \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \tag{73}$$

It is convenient to write these matrices in terms of 2 x 2 submatrices as follows:

$$P_1 = \begin{vmatrix} \pi_{11} & 0 \\ \pi_{12} & \pi_{23} \end{vmatrix} \qquad P_2 = \begin{vmatrix} 0 & \pi_{21} \\ 0 & 0 \end{vmatrix} \tag{74}$$

$$P_1^{-1} = \begin{vmatrix} \pi_{11}^{-1} & 0 \\ -\pi_{13}^{-1}\,\pi_{12}\,\pi_{11}^{-1}\,, & \pi_{13}^{-1} \end{vmatrix} \tag{75}$$

and

$$\pi_{11} = \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix};\ \pi_{12} = \pi_{11} \begin{vmatrix} g_1 & 0 \\ 0 & g_2 \end{vmatrix};\ \pi_{13} = \begin{vmatrix} 1 & 1 \\ \lambda_3 & \lambda_4 \end{vmatrix} \begin{vmatrix} (\lambda_3 + i\gamma_o)^2 & 0 \\ 0 & (\lambda_4 + i\gamma_o)^2 \end{vmatrix}; \tag{76}$$

$$\pi_{11}^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{vmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{vmatrix};\quad \pi_{12}^{-1} = \begin{vmatrix} g_1^{-1} & 0 \\ 0 & g_2^{-1} \end{vmatrix}; \tag{77}$$

$$\pi_{13}^{-1} = \frac{1}{\lambda_4 - \lambda_3} \begin{vmatrix} (\lambda_3 + i\gamma_o)^2 & 0 \\ 0 & (\lambda_4 + i\gamma_o)^2 \end{vmatrix} \begin{vmatrix} \lambda_4 & -1 \\ -\lambda_3 & 1 \end{vmatrix}; \tag{78}$$

$$\pi_{21} = \begin{vmatrix} 1 & 1 \\ \lambda_3 & \lambda_4 \end{vmatrix};\quad \pi_{21}^{-1} = \frac{1}{\lambda_4 - \lambda_3} \begin{vmatrix} \lambda_4 & 1 \\ -\lambda_3 & 1 \end{vmatrix}. \tag{79}$$

Hence,

$$P = P_1\left[1 + y^2 \gamma_o^2 P_1^{-1} P_2\right]$$

$$P^{-1} = \left[1 + y^2\gamma_o^2 P_1^{-1} P_2\right]^{-1} P_1^{-1} = P_1^{-1} + \phi\,(y^2), \qquad (80)$$

where $$\phi(y^2) = \sum_1^{\infty} (-1)^n \,(y^2\gamma_o^2)^n\, (P_1^{-1}P_2)^n P_1^{-1}.$$

The expansion is valid and uniformly convergent as long as

$$\left| 4\, y^2\, \gamma_o^2\, M \right| < 1 \qquad (81)$$

where M is the least upper bound of the elements of P_1^{-1} P_2. Note that the elements of P_1^{-1} and P_2 are bounded for all values of y except that of π_{11}^{-1}. The elements of π_{11}^{-1} are $O(\frac{1}{y})$. Hence,

$$M = O(\tfrac{1}{y}) \qquad (82)$$

and $$\left| 4\, y^2\, \gamma_o^2\, M \right| = O(y)$$

Thus by taking y small enough, the inequality (81) will be satisfied.

The matrix R of which the matrizant is sought in equation (63) may now be expanded in powers of y^2 as follows:

$$\Lambda = P^{-1} A_1\, P = \text{diag.}\, (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

$$R_1 = P^{-1} A_{2x}\, P = P_1^{-1} A_{2x}\, P_1 + \Psi\,(y^2, V_{oz})\,, \qquad (83)$$

where $$\Psi\,(y^2, V_{oz}) = \sum_1^{\infty} (-1)^n \,(y^2\gamma_o^2)^n\, (P_1^{-1} P_2)^{n-1}\left[P_1^{-1} P_2\, P_1^{-1} A_{2x}\, P_1 - P_1^{-1} A_{2x}\, P_1\right]\;; \qquad (84)$$

$$R_2 = -P^{-1} P_x = -P_1^{-1} P_{1x} - \theta\,(y^2, V_{oz})\,,$$

where
$$\theta = \sum_{1}^{\infty} (-1)^n \, (y^2 \gamma_o^{\,2})^n \, (P_1^{-1} P_2)^{n-1} \left[P_1^{-1} P_2 P_1^{-1} P_{1x} - 3 \frac{\gamma_{ox}}{\gamma_o} P_1^{-1} P_2 - P_1^{-1} P_{2x} \right] . \qquad (85)$$

Thus,

$$R = \Lambda + R_1 + R_2 = \Lambda + P_1^{-1} A_{2x} P_1 - P_1^{-1} P_{1x} + \Psi(y^2, V_{oz}) - \theta\,(y^2, V_{oz}). \qquad (86)$$

It has been shown(10) that it is possible to express the solution of the differential equation as a sum of exponential functions by letting

$$R = D + R_3 \qquad (87)$$

D is a diagonal matrix $=$ diag. (d_n)

Then using equations (57) and (58),

$$\begin{aligned} \Omega\{R\} &= \Omega\{D\}\,\Omega\{M\} \\ M &= \Omega^{-1}\{D\}\; R_3\, \Omega\{D\} \end{aligned} \qquad (88)$$

$$\Omega\{D\} = \text{diag.}\left\{ \exp \int_{z_o}^{z} d_m \, d\,x \right\}, \quad m = 1,2,3,4. \qquad (89)$$

Now let the elements of these matrices be chosen as follows:

$$d_{mm} = \left[\Lambda + P_1^{-1} A_{2x} P_1 - P_1^{-1} P_{1x} \right]_{mm} \qquad (90)$$

$$(R_3)_{mn} = (1 - \delta_{mn}) \left[P_1^{-1} A_{2x} P_1 - P_1^{-1} P_{1x} \right]_{mn} + \Psi_{mn} - \theta_{mn} \qquad (91)$$

$$M_{mn} = (R_3)_{mn} \exp \left[\int_{z_o}^{z} (d_m - d_n) \, d\,x \right]. \qquad (92)$$

Also let $\Omega\{M\} = 1 + \Psi$

$$\Psi_{mn} = \int_{z_o}^{z} M_{mn} \, d\,x + \sum_{k} \int_{z_o}^{z} M_{mk}(x_2) \, d\,x_2 \int_{z_o}^{x_2} M_{kn}(x_1) \, dx_1 + \cdots\cdots \qquad (93)$$

Ψ_{mn} is the sum of two terms, of which one is a combination of the derivatives of V_o and the other is a power series in y^2. The solution of the differential equation is now given by

$$U = P \left\{ D \right\} \left[P_1^{-1}(z_o) + \Psi P_1^{-1}(z_o) + \phi(z_o) + \Psi \phi(z_o) \right] U(z_o). \qquad (94)$$

The elements of U are

$$U_m = \sum_j \left[\sum_k \left(P_1^{-1}(z_o) + \Psi P_1^{-1}(z_o) + \phi(z_o) + \Psi \phi(z_o) \right)_{jk} U_k(z_o) \right] P_{mj} \exp \int_{z_o}^{z} djdx \qquad (95)$$

Thus the following theorem has been proved.

Theorem III

The solution of the differential equation (44) may be expressed as the sum of four exponential functions with coefficients given in the form of uniformly absolutely convergent series in the region $z_o \leq z \leq z_1$ in powers of y^2 and V_{oz}.

V. SUMMARY OF RESULTS

The uniqueness theorem for the electromagnetic field quantities in charge free space has been extended under certain conditions to the field quantities in an electron beam. The uniqueness theorem states that if J and V are parallel within a multiply connected region G, the sinusoidal field quantities which are bounded and possess continuous first derivatives are determined uniquely within G by specification on the boundary surfaces of the component of E or the component of H tangent to the boundary surfaces and the velocity or the component of J normal to the boundary surfaces. This uniqueness theorem is utilized to obtain the boundary conditions which determine a unique solution for the field quantities in a right cylinder completed filled with an electron beam. The theorem for the required boundary conditions states that if V_o is an analytic function of the axial dimension only and $H_o = \infty$, the fields within the right cylinder exist and are uniquely determined by the boundary condition that the value of J, J_z or V, and the components of E and H in the transverse plane be given on a transverse plane. J and the axial component of E are determined by a system of two second-order ordinary differential equations, which are coupled to each other. This system of differential equations is solved by the matrizant method, and the solution is expressed as the sum of four exponential functions with coefficients given in the form of uniformly absolutely convergent series in powers of γ^2 and V_{oz}.

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